Fluctuation-induced collective motion: A single-particle density analysis

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In a system of noisy self-propelled particles with interactions that favor directional alignment, collective motion will appear if the density of particles increases beyond a certain threshold. In this paper, we argue that such a threshold may depend also on the profiles of the perturbation in the particle directions. Specifically, we perform mean-field, linear stability, perturbative, and numerical analyses on an approximated form of the Fokker-Planck equation describing the system. We find that if an angular perturbation to an initially homogeneous system is large in magnitude and highly localized in space, it will be amplified and thus serves as an indication of the onset of collective motion. Our results also demonstrate that high particle speed promotes collective motion.

I. INTRODUCTION

The interesting phenomena of flocking in animals [1–3] and self-organized patterns in motile cells [4–6] are currently driving the intense theoretical study of collective motion (CM) among self-propelled particles [7–17]. In particular, a comprehensive linear stability analysis on the onset of collective motion from the perspective of Boltzmann equation has recently appeared [18]. Models for collective motion usually involve motile particles that possess alignment interactions and angular noise. Collective motion is then observed if the density of particles increases beyond a certain threshold. Besides density fluctuations, fluctuations in the heading directions of the particles constitute another important aspect of the system. Here, we study a minimal model for collective motion and show that the threshold for collective motion transition may depend on the profiles of directional fluctuations. Specifically, we find that an initial directional perturbation to a spatially homogeneous system will be amplified, if the perturbation is large in magnitude and is highly localized in space. We also demonstrate that high particle speed promotes collective motion.

To achieve our results, we first write down the Fokker-Planck equation describing the single-particle density distribution of the system in Sec. II. We then investigate in Sec. III the equation in the Fourier space of the directional component, and argue that only the lower order modes are important at the onset of collective motion. As a result, the dynamics of the distribution function can be captured by a set of three nonlinear coupled differential equations, which we subsequently study with linear stability analysis in Sec. IV. In Sec. V, we go beyond the linear stability regime by investigating the dynamical equations perturbatively and numerically.

II. MODEL

In this work, we follow [16] and consider a minimal model for collective motion in two dimensions, where every particle is assumed to have constant speed and that their interactions consist only of directional alignment mechanism. Noise is incorporated in the direction of travel. Specifically, let there be N particles in a volume of V; their equations of motion are

$$\frac{d\mathbf{r}_i}{dt} = \frac{2u}{\pi}\mathbf{v}(\theta_i) \tag{1}$$

$$\frac{d\theta_i}{dt} = -\frac{\partial U}{\partial \theta_i}(\mathbf{R},\Theta) + \sqrt{2D}\,\eta_i(t) \tag{2}$$

where $1 \le i \le N$, $\mathbf{R} \equiv (\mathbf{r}_1,\ldots,\mathbf{r}_N)$, $\Theta \equiv (\theta_1,\ldots,\theta_N)$, $\mathbf{v}(\theta) \equiv (\cos\theta,\sin\theta)$, and the noise is assumed to be Gaussian characterized by the following moments:

$$\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t-t'). \tag{3}$$

Moreover, the alignment interaction is assumed to be of very short range and thus can be approximated by a delta function,

$$U(\mathbf{R},\Theta) = -\frac{g}{\pi}\sum_{i<j}\delta^{(2)}(\mathbf{r}_i-\mathbf{r}_j)\cos(\theta_i-\theta_j). \tag{4}$$

If we denote the probability distribution of the density of particles in the state $(\mathbf{R},\Theta)$ at time t by $f(t,\mathbf{R},\Theta)$, then the Fokker-Planck equation corresponding to the system is [19]

$$\frac{\partial f}{\partial t} = \sum_i \left\{ D\frac{\partial^2}{\partial\theta_i^2}f - \frac{2u}{\pi}\nabla_{\mathbf{r}_i}\cdot[\mathbf{v}(\theta_i)f] \right\}$$
$$+ \frac{g}{\pi}\sum_{i<j}\frac{\partial}{\partial\theta_i}[\delta^{(2)}(\mathbf{r}_i-\mathbf{r}_j)\sin(\theta_i-\theta_j)f]. \tag{5}$$

Naturally, we are not interested in all the information captured by f, and we will from now on focus on the single-particle density function, ρ,

$$\rho(\mathbf{r}_1,\theta_1) = \frac{(N!)\int dr_2\cdots dr_N d\theta_2\cdots d\theta_N f(\mathbf{R},\Theta)}{(N-1)!}. \tag{6}$$

From Eq. (5), we can express ρ in terms of the two-particle density function $\rho^{(2)}$,

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$$\frac{\partial \rho(\mathbf{r},\theta)}{\partial t} = D\frac{\partial^2 \rho(\mathbf{r},\theta)}{\partial \theta^2} - \frac{2u}{\pi}\left[\cos\theta\frac{\partial \rho(\mathbf{r},\theta)}{\partial x} + \sin\theta\frac{\partial \rho(\mathbf{r},\theta)}{\partial y}\right]$$

$$+ \frac{g}{\pi}\frac{\partial}{\partial \theta}\left[\left[\int d\theta' \int dr' \,\delta^{(2)}(\mathbf{r}-\mathbf{r}')\sin(\theta-\theta')\right.\right.$$

$$\left.\left.\times \rho^{(2)}(\mathbf{r},\theta,\mathbf{r}',\theta')\right]\right]. \tag{7}$$

where

$$\rho^{(2)}(\mathbf{r}_1,\theta_1,\mathbf{r}_2,\theta_2) = \frac{(N!)\int dr_3\cdots dr_N d\theta_3\cdots d\theta_N f(\mathbf{R},\Theta)}{(N-2)!}. \tag{8}$$

The above manipulation is akin to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy formalism [20]. To continue with our analytical treatment, we will ignore the second ordered correlation and adopt the *product distribution* assumption: $\rho^{(2)}(\mathbf{r},\theta,\mathbf{r}',\theta') = \rho(\mathbf{r},\theta)\rho(\mathbf{r}',\theta')$. This assumption is similar to the molecular chaos assumption in the context of Boltzmann equation, and is also adopted in [15,18].

By Fourier transforming the above equation with respect to the angular variable, θ, we have

$$\partial_t \hat{\rho}_n(\mathbf{r}) = -Dn^2 \hat{\rho}_n(\mathbf{r}) - u[\partial_x(\hat{\rho}_{n+1}(\mathbf{r}) + \hat{\rho}_{n-1}(\mathbf{r}))$$

$$+ i\partial_y(\hat{\rho}_{n-1}(\mathbf{r}) - \hat{\rho}_{n+1}(\mathbf{r}))]$$

$$- gn[\hat{\rho}_{-1}(\mathbf{r})\hat{\rho}_{n+1}(\mathbf{r}) - \hat{\rho}_1(\mathbf{r})\hat{\rho}_{n-1}(\mathbf{r})] \tag{9}$$

where $\rho(\mathbf{r},\theta) = \sum_{n=-\infty}^{\infty}\hat{\rho}_n(\mathbf{r})e^{-in\theta}$.

Since $\hat{\rho}_n(\mathbf{r})$ are complex, we will denote them by $a_n(\mathbf{r}) + ib_n(\mathbf{r})$ where a_n and b_n are real functions. In relation to the original density function, we have

$$\rho(\mathbf{r},\theta) = \sum_{n\in\mathbf{Z}}[a_n(\mathbf{r}) + ib_n(\mathbf{r})]e^{-in\theta}$$

$$= a_0(\mathbf{r}) + 2\sum_{n>1}[a_n(\mathbf{r})\cos(n\theta) + b_n(\mathbf{r})\sin(n\theta)] \tag{10}$$

where for the second equality, the following conditions for the a_n and b_n have been employed:

$$a_n = a_{-n}, \quad b_n = -b_{-n}, \tag{11}$$

which are due the fact that ρ is real. Writing Eq. (9) in terms of the a_n and b_n, we have for $n \in \mathbf{Z}$,

$$\partial_t a_n = -Dn^2 a_n - u[\partial_x(a_{n+1} + a_{n-1}) - \partial_y(b_{n-1} - b_{n+1})]$$

$$- gn[a_1(a_{n+1} - a_{n-1}) + b_1(b_{n-1} + b_{n+1})] \tag{12}$$

$$\partial_t b_n = -Dn^2 b_n - u[\partial_x(b_{n+1} + b_{n-1}) + \partial_y(a_{n-1} - a_{n+1})]$$

$$- gn[a_1(b_{n+1} - b_{n-1}) - b_1(a_{n+1} + a_{n-1})]. \tag{13}$$

Note that the arguments $(t,\mathbf{r})$ in a_n and b_n are omitted in the above equations to ease notation.

III. MEAN-FIELD APPROXIMATION

To avoid having to deal with the above infinite set of differential equations, we will sort to truncate the number of

differential equations to be considered. To do so, we first study the system in a mean-field manner [21]; i.e., we set $\hat{\rho}_n(t,\mathbf{r}) = \hat{\rho}_n(t)$ for all $\mathbf{r}$. Equations (12) and (13) then become

$$\frac{da_n}{dt} = -Dn^2 a_n - gn[a_1(a_{n+1} - a_{n-1}) + b_1(b_{n-1} + b_{n+1})] \tag{14}$$

$$\frac{db_n}{dt} = -Dn^2 b_n - gn[a_1(b_{n+1} - b_{n-1}) - b_1(a_{n+1} + a_{n-1})]. \tag{15}$$

Note that $da_0/dt = 0$ due to the fact that a_0 corresponds to the overall density of the system, which does not change.

Let us assume that the b modes are not excited at $t = 0$ and so we need only focus on the a modes. By inspecting Eq. (10), we see that the omission of the b modes is the same as focusing only on angular perturbation of the form $\cos(\theta)$, i.e., the particles are more likely to be heading in the positive x direction. With this simplification, the first three modes are of the form

$$\frac{da_1}{dt} = ga_0 a_1 - (Da_1 + ga_1 a_2) \tag{16}$$

$$\frac{da_2}{dt} = 2ga_1^2 - (4Da_2 + ga_1 a_3) \tag{17}$$

$$\frac{da_3}{dt} = 3ga_1 a_2 - (9Da_3 + ga_1 a_4). \tag{18}$$

At the onset of CM from a spatially and angularly homogeneous system, we expect that $|a_n| \ll 1$ for $n > 1$. Let us define ϵ as $\max_{n>1}|a_n|$ at the onset of CM, we see that only da_1/dt is of order ϵ while all the time derivatives for the higher order modes are of order ϵ^2. Furthermore, the coefficients associated with the damping term D for the n-th modes scale with n^2, which further suggests that only the lower order modes are important. Another corroborating evidence is from [15,18] where the authors employed their *scaling ansatz*, which is supported by their numerical simulations, to argue that the higher order modes are indeed negligible at the onset of CM. Based on all these reasons, we will truncate the original dynamical equations, Eq. (9), by omitting all $\hat{\rho}_n$ for $n > 2$. Focusing again only on the a modes, we have

$$\frac{da_0}{dt} = 0 \tag{19}$$

$$\frac{da_1}{dt} = (-D + ga_0 - ga_2)a_1 \tag{20}$$

$$\frac{da_2}{dt} = -4Da_2 + 2ga_1^2. \tag{21}$$

A simple fixed-point analysis on the above equations indicate that the existence of nonzero fixed point for a_1 and a_2 is only possible when

$$ga_0 - D > 0. \tag{22}$$

This condition has previously been derived in [16]. Expectedly, the above condition indicates that collective motion is facilitated by having strong interaction (g), high particle density (a_0) and weak noise (D).

IV. LINEAR STABILITY ANALYSIS

We now continue with our truncation approximation, but with the spatial variable reinstalled into Eqs. (14) and (15). Before we start to analyze the set of differential equations, we note that by inspecting Eqs. (12) and (13), we see that the $a_{n \geq 1}$ and $b_{n \geq 1}$ modes are coupled exclusively to different spatial dimensions–the x and y dimensions respectively. In other words, if the system is initially homogeneous in the x dimension, then the x dimension will remain homogeneous, and vice versa. We will therefore, as in the previous section, assume that the b modes are not excited and focus only on the a modes. With this simplification, we arrive at the following dynamical equations:

$$\partial_t \alpha = -2u\partial_x \beta \tag{23}$$

$$\partial_t \beta = -D\beta - u\partial_x(\alpha + \gamma) + g\beta(\alpha - \gamma) \tag{24}$$

$$\partial_t \gamma = -4D\gamma - u\partial_x \beta + 2g\beta^2, \tag{25}$$

where we have used the Greek letters α, β, and γ to denote a_0, a_1, and a_2, respectively.

The fixed point in the homogeneous phase corresponds to $\alpha = 1$, $\beta = 0$, and $\gamma = 0$ where we have set the unit length in such a way that the density of the particles is one. We now perform linear stability analysis on this fixed point by considering the linear response of the system to a small perturbation of the form

$$\alpha = 1 + A e^{\lambda t + iqy} \tag{26}$$

$$\beta = B e^{\lambda t + iqy} \tag{27}$$

$$\gamma = C e^{\lambda t + iqy} \tag{28}$$

where $A, B, C \ll 1$ and q is an arbitrary frequency. Substituting the above into Eqs. (23) and (24) gives the following condition on λ:

$$\frac{\lambda(\lambda + D - g)(\lambda + 4D)}{3\lambda + 8D} = -u^2 q^2, \tag{29}$$

which indicates that $\lambda > 0$ if and only if $g > D$. In other words, we have recovered the condition found in our previous mean-field analysis [cf. Eq. (22)].

Although this result is consistent with what we found in the previous section, pieces of the picture at the onset of CM are still lacking. For instance, the phase-transition condition found here does not depend on the speed of the particle u. This is unsatisfactory because we know that long-range order would not be possible if $u=0$ [8,9]. Moreover, it is desirable to see how the coupling between the spatial and temporal dimension affects the rise of the excited mode β. To gain

insight in these questions, we will go beyond the linear stability regime and analyze the dynamical equations with perturbative method in the next section.

V. BEYOND LINEAR STABILITY ANALYSIS

In this section, we will study Eqs. (23)–(25) perturbatively. Specifically, we will assume that $D, g \ll 1$, in the units of distance and time set by having $\alpha(t=0, x)=1$ and $u = 1/\sqrt{3}$. Physically, these assumptions amount to limiting our discussion to the regime where the particles' angular fluctuations and interaction strength are small. Furthermore, since we are primarily interested in the dynamics at the onset of CM, we will assume that g/D is of order unity [cf. Eq. (22)]. These assumptions allow us to employ D (or equivalently, g) as the expansion parameter in our perturbative treatment. In contrast to the linear stability analysis in the previous section where the perturbation magnitude is assumed to be small enough that the nonlinear term is negligible, the perturbative approach adopted here allows us to study the effects of the nonlinear term on the dynamics.

In the aforementioned units, Eqs. (23)–(25) are:

$$\partial_t \alpha = -\frac{2}{\sqrt{3}} \partial_x \beta \tag{30}$$

$$\partial_t \beta = -D\beta - \frac{1}{\sqrt{3}} \partial_x(\alpha + \gamma) + g\beta(\alpha - \gamma) \tag{31}$$

$$\partial_t \gamma = -4D\gamma - \frac{1}{\sqrt{3}} \partial_x \beta + 2g\beta^2. \tag{32}$$

We now expand α as

$$\alpha = \alpha_0 + D\alpha_1 + \mathcal{O}(D^2), \tag{33}$$

and similarly for β and γ. The zeroth order (in D) terms follow the following differential equations:

$$\partial_t \alpha_0 = -\frac{2}{\sqrt{3}} \partial_x \beta_0 \tag{34}$$

$$\partial_t \beta_0 = -\frac{1}{\sqrt{3}} \partial_x(\alpha_0 + \gamma_0) \tag{35}$$

$$\partial_t \gamma_0 = -\frac{1}{\sqrt{3}} \partial_x \beta_0. \tag{36}$$

The above set of differential equations can be solved by employing the Laplace-Fourier transform method. For the initial conditions of $\alpha_0(t=0, x)=1$, $\beta_0(t=0, x)=\xi e^{-y^2/(2\sigma^2)}/\sqrt{2\pi}\sigma$, and $\gamma_0(t=0, x)=0$. The solutions are

$$\alpha_0 = 1 + \frac{\xi}{\sqrt{6\pi}\sigma}[U^- - U^+] \tag{37}$$

$$\beta_0 = \frac{\xi}{2\sqrt{2\pi}\sigma}[U^- + U^+] \tag{38}$$

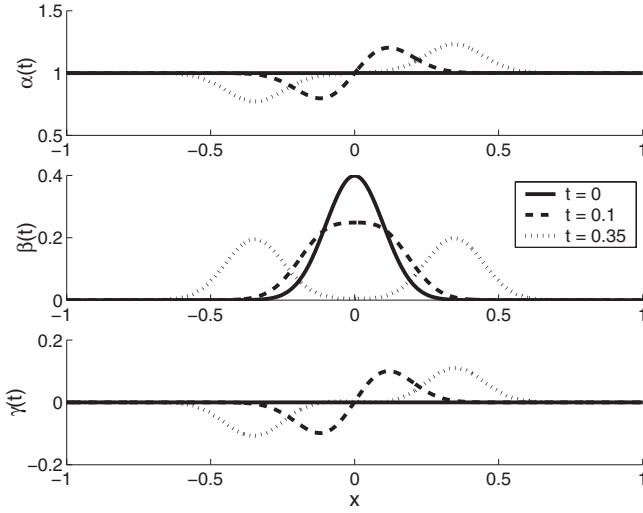FIG. 1. The density profiles of α, β and γ at different times obtained by numerically solving the set of differential equations in Eqs. (30)–(32), with the following parameters: $u=1/\sqrt{3}$, and $D=g=\xi=\sigma=0.1$.

$$\gamma_0 = \frac{\xi}{2\sqrt{6\pi}\sigma}[U^- - U^+], \tag{39}$$

where

$$U^{\pm} = \exp\left(-\frac{(x \pm t)^2}{2\sigma^2}\right). \tag{40}$$

In other words, a Gaussian perturbation in β at $t=0$ splits into two Gaussian distributions traveling in opposite directions with unit speed (as a result of setting u to $1/\sqrt{3}$). The perturbation also induces in α and γ two solitary waves in the form a Gaussian distribution traveling with unit speed in the positive direction, and an inverted Gaussian density wave traveling in the opposite direction (cf. Fig. 1). This is akin to the stripe traveling-wave pattern found in the CM phase [10,18].

The differential equations governing the first-order terms are

$$\partial_t \alpha_1 = -\frac{2}{\sqrt{3}} \partial_x \beta_1 \tag{41}$$

$$\partial_t \beta_1 = -\beta_0 - \frac{1}{\sqrt{3}} \partial_x(\alpha_1 + \gamma_1) + \frac{g}{D} \beta_0(\alpha_0 - \gamma_0) \tag{42}$$

$$\partial_t \gamma_1 = -4\gamma_0 - \frac{1}{\sqrt{3}} \partial_x \beta_1 + \frac{2g}{D} \beta_0^2. \tag{43}$$

where α_0, β_0, and γ_0 above are now given by Eqs. (37)–(39). The initial conditions for the above equations are: $\alpha_1(t=0,x)=\beta_1(t=0,x)=\gamma_1(t=0,x)=0$.

Before we attempt to study the above set of differential equations, let us look for some meaningful quantities that quantifies the effect of the initial perturbation. For instance, the temporal evolution of total increase in β_1 due to the initial perturbation can be obtained from Eq. (42),

$$\partial_t\left(\int_{-\infty}^{\infty} dy\, \beta_1(t,y)\right) = \int_{-\infty}^{\infty} dy\left(\frac{g}{D}\alpha - 1\right)\beta = \xi\left(\frac{g}{D} - 1\right) \tag{44}$$

This indicates that when summed over the whole space, the β mode is amplified only if $g>D$. This is again consistent with the result we obtained in Secs. III and IV.

Besides the above quantity, the following two quantities are also of interest:

$$A(t) \equiv \int_0^{\infty} dx\, \alpha(t,x); \quad B(t) \equiv \int_0^{\infty} dx[\beta(t,x) - \beta(0,x)]. \tag{45}$$

Namely, A and B correspond to the responses in the density (α) and in the β mode *in the direction* of the angular perturbation. These are in fact arguably better quantities to consider as they capture the directional nature of the perturbation. From Eq. (42), we have

$$\frac{dB(t)}{dt} = \frac{\xi(g-D)}{2} + \frac{g\xi^2}{8\sqrt{3\pi}\sigma}\text{erf}\left(\frac{t}{\sigma}\right) \tag{46}$$

$$+ \frac{D}{\sqrt{3}}[\alpha_1(t,x=0) + \gamma_1(t,x=0)]. \tag{47}$$

In the Appendix, we demonstrate that

$$\alpha_1(t,x=0) + \gamma_1(t,x=0) = \frac{g\xi^2}{\pi\sigma^2}t + \mathcal{O}(t^3). \tag{48}$$

In other words, up to order $\mathcal{O}(t^3)$, we have

$$B(t) = \frac{\xi(g-D)t}{2} + \frac{5gt^2}{8\sqrt{3}\pi}\left(\frac{\xi}{\sigma}\right)^2. \tag{49}$$

The third term in the right-hand side above highlights the importance of the term ξ/σ, especially when $D \simeq g$. Figures 2(a) and 2(b) display the temporal evolutions of $A(t)$ and $B(t)$ by solving Eqs. (30) and (31) numerically in the case of $D=g$. They clearly show the amplification of the initial perturbation, which we have taken as an indication for the onset of collective motion in longer time. Figure 2(c) demonstrates that at short time, the dynamics is well described by the expression in Eq. (49). Furthermore, due to the positive second term in the R.H.S. above, the formula for $B(t)$ suggests that there is a possibility of perturbation amplification even if $D>g$, e.g., when $\xi/\sigma \gg 1$. This is indeed shown to be the case in Figs. 3(a) and 3(b), where amplification of the perturbation is seen for $D/g \sim 1.1$. In other words, a sharp perturbation in the angular domain is able to induce collective motion even if the density is below the phase-transition threshold as obtained in the mean-field model.

If we now restore the speed, u, and the initial density, c, where $c=\alpha(t=0,x)$, into Eq. (49), we have

$$B(t) = \frac{\xi(cg-D)t}{2} + \frac{5gut^2}{8\pi}\left(\frac{\xi}{\sigma}\right)^2. \tag{50}$$

Note that the speed of the particles only appears in the second term above, which is positive. Hence, the above formula
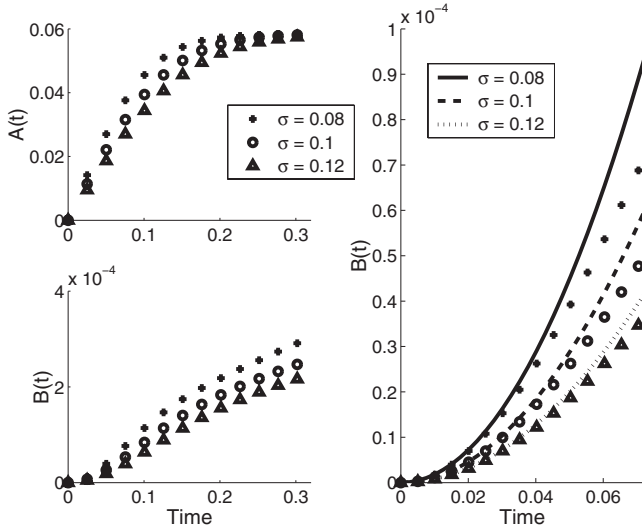
FIG. 2. The temporal evolutions of (a) A [cf. Eq. (45)] and (b) B [cf. Eq. (45)], obtained by numerically solving the set of differential equations in Eqs. (30) and (32), with $D=g=\xi=0.1$. (c) The zoom-in plot of $B(t)$ at small time with the three curves corresponding to the theoretical expressions given in Eq. (49).

suggests that the particle speed has a net effect of amplifying the initial perturbation and thus facilitating collective motion transition.

As verification on the validity of our approximation adopted in our analytical calculations, we numerically simulate Eqs. (12) and (13) with higher order modes included and find that there are no discernible differences for the parameter range investigated in this work (cf. Fig. 4).

VI. CONCLUSION

In summary, starting with a Fokker-Planck equation for a minimal model of CM, we derived a set of three coupled
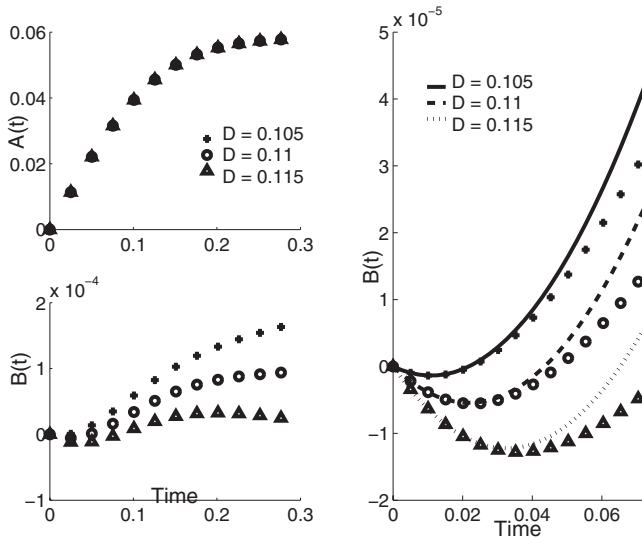

FIG. 3. The temporal evolutions of (a) A [cf. Eq. (45)] and (b) B [cf. Eq. (45)], obtained by numerically solving the set of differential equations in Eqs. (30) and (32), with $g=b=\sigma=0.1$. The zoom-in plot of $B(t)$ at small time with the three curves corresponding to the theoretical expressions given in Eq. (49).
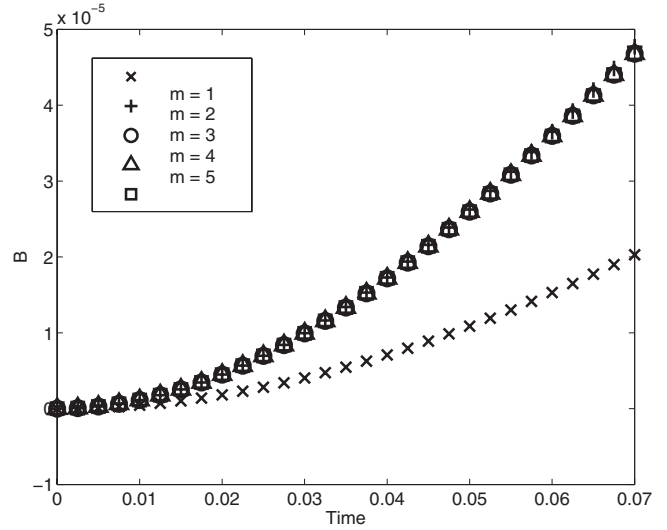

FIG. 4. The evolution of $B(t)$ obtained by numerical simulating the differential equations Eqs. (12) and (13) with the initial condition discussed in Sec. V. The parameters are $g=D=\xi=\sigma=0.1$. In the simulations, only the 0th to the mth a modes are included. In other words, we assume that $a_{n>m}=0$ and $b_{n\in\mathbf{Z}}=0$.

differential equations that describes the system at the onset of CM. We then studied the equations with mean-field, linear stability, perturbative, and numerical analyses and found that if an angular perturbation is large in magnitude and highly localized in space, it will be amplified and thus serves as an indication of the onset of collective motion. Our calculations also demonstrate the importance of particle speed for collection motion transition. As a result, it is indicative that the critical point for CM may depend on the speed u, the perturbation magnitude b and the perturbation wavelength σ. This is in contrast to the mean-field and linear stability analyses where only the hydrodynamic, or infinite-wavelength, mode dictates the onset of CM. Our results therefore highlight the importance of incorporating the nonlinear term into the analysis.

The main limitation of this work is on the approximation adopted—the omissions of higher order modes. While we believe that such an approximation is appropriate at the onset of CM, it would be highly desirable to have a systematic method to incorporate the higher order modes into the dynamics. Besides the consideration of the higher order modes, singular perturbation method would also be needed to investigate the long-time behavior of the system [22]. We believe that these aspects would constitute two promising directions for future investigation.

APPENDIX: SOLUTIONS OF THE PERTURBATION PROBLEM TO FIRST ORDER

We are unable to solve the set of differential equations shown in Eqs. (41)–(43) analytically. But since only the leading orders in x and t are of interests, we will replace the $U^{\pm}$ in $\alpha_0, \beta_0, \gamma_0$ [cf. Eqs. (37)–(39)] by

$$\widetilde{U}^{\pm} \equiv 1 - \frac{(x \pm t)^2}{2\sigma^2} + \left[\frac{(x \pm t)^2}{2\sigma^2} \right]^2. \qquad (A1)$$

With this simplification, the differential equations can be solved by the Laplace-Fourier transform method and the relevant results are

$$\alpha_1(t, x = 0) = \mathcal{O}(t^3), \quad \gamma_1(t, x = 0) = \frac{g\xi^2}{\pi\sigma^2}t + \mathcal{O}(t^3).$$

$$(A2)$$

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